Matrix Algebra 3 (Loose ends)

:Trace

Definition: Let $A \in \mathbb{R}^{mxm}$. The *trace* of A is the sum of its diagonal components, i.e.

$$tr(A) \equiv \sum_{i=1}^{m} a_{ii}$$

Properties:

1. $(\forall c \in \mathbb{R})\ \ tr(cA) = c \bullet tr(A);\ tr(A+B) = tr(A) + tr(B)$
2. $tr(A) = tr(A')$
3. if both products exist, $tr(AB) = tr(BA)$
4. if A is idempotent, $tr(A) = rk(A)$

:Partitioned Matrices

We can partition a matrix into submatrices. For example, if

$$A = \begin{bmatrix} 4 & 0 & 2 & -1 \\ 6 & 5 & 1 & 1 \\ -3 & 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 4 & 0 & 2 \\ 6 & 5 & 1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -3 & 2 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 5 \end{bmatrix}$$

We require that the submatrices appearing in any given row (column) must themselves have all the same number of rows (columns).

The usual rules for the addition and multplication of matrices apply directly to submatrices *provided the operations are always applied to conformable submatrices*.

- If A and B are two partitioned matrices, then

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

provided that all sums $A_{ij} + B_{ij}$ exist.

- Similarly,

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} \end{bmatrix}$$

- In particular, if we write $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ where $X_i \in \mathbb{R}^{nxK_i}$, then

$$X'X = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_1 \end{bmatrix}$$

and

$$(X'X)^{-1}X'X_1 = \begin{bmatrix} I_{K_1} \\ 0 \end{bmatrix}$$

:Quadratic Forms

Definition: Suppose $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{mxm}$ with A symmetric. Then $x'Ax$ is called a *quadratic form*.

Rks:

- $x'Ax \in \mathbb{R}$
- It is not restrictive to assume that A is symmetric. Using $x'Ax = x'A'x = x'\widetilde{A}x$ where $\widetilde{A} = (A + A')/2$, we can always replace A with $\widetilde{A}$.
- For $m = 2$,

$$x'Ax = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

Definition: The matrix A is called *positive definite (p.d.)* if $x'Ax > 0$ for $\forall x \neq 0$.

Definition: The matrix A is called *nonnegative definite (n.n.d)* if $x'Ax \geq 0$ for $\forall x$.

Definition: The matrix A is called *positive semidefinite (p.s.d.)* if $x'Ax \geq 0$ for $\forall x$ and for some $x \neq 0$ we have $x'Ax = 0$.

Rks:

- By changing the sign on the inequalities, we can define the set of *negative definite*, *nonpositive definite*, and *negative semidefinite* matrices.
- Many (most?) authors do not distinguish between nonnegative definite and positive semidefinite (or nonpositive definite and negative semidefinite matrices).

Theorems:

1. A is positive definite iff we can write $A = RR'$ for some nonsingular matrix R.
2. A is nonnegative definite iff we can write $A = RR'$ for some matrix R.
3. If A is positive definite, and B has full column rank, then $B'AB$ is positive definite.

Definition: If we can write $A = RR'$ for some matrix R, then R is called a *square-root matrix* of A.

Rks:

- If A has a square-root, then $x'Ax = x'RR'x = c'c$ where $c \equiv R'x$.
- There are lots ($2 \cdot m!$) square root matrices. If R is a square root, then so is $\widetilde{R} = R\Theta$ for any orthogonal matrix Θ (i.e. $\Theta^{-1} = \Theta'$).

Definition: If $A, B \in R^{mxm}$, we say "A is bigger than B (in the matrix sense)" , written $A \geq B$, if $A - B$ is n.n.d.

Rk: The matrix $\geq$ is a partial ordering.

Theorem: If A, B are invertible, then $A \geq B$ iff $B^{-1} \geq A^{-1}$.

Definition: For any $c \in \mathbb{R}$, $\{x : x'Ax = c\}$ is called a *level curve*.

Rks:

- For $m = 2$, if A is p.d., the level curves form ellipses.
- For $m > 2$, if A is p.d., the level curves form ellipsoids.

:Spectral Representation

Theorem: Every symmetric matrix $A \in \mathbb{R}^{mxm}$ can be expressed as

$$A = P \Lambda P'$$

where $\Lambda \in \mathbb{R}^{mxm}$ is a diagonal matrix, i.e.

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \equiv diag(\lambda_i)$$

and $P \in \mathbb{R}^{mxm}$ is an orthogonal matrix (i.e. $PP' = I$).

Rks:

- The λ_i are called *eigenvalues* and the corresponding columns P_i are called *eigenvectors*.

Properties:

1. A is p.d. (n.n.d.) iff each of its eigenvalues are positive (nonnegative)
2. A choice for the square-root matrix of A is given by $A^{1/2} \equiv Pdiag(\lambda_i^{1/2})P'$.
3. $A^m = Pdiag(\lambda_i^m)P'$ for all integers m.
4. If A is symmetric and idempotent, then all its eigenvalues are either 0 or 1.